UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

MODULE: **ASMA2B2**

CAMPUS: **APK**

EXAM: **JUNE 2014**

DATE: **31 MAY 2014** **SESSION: 8:30-11:00**

ASSESSOR(S): **C MARAIS**

INTERNAL MODERATOR: **W MORTON**

DURATION: **2 HOURS** **MARKS: 60**

SURNAME AND INITIALS:

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STUDENT NUMBER:

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CONTACT NR:

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NUMBER OF PAGES: 11 PAGES

INSTRUCTIONS: **ANSWER ALL THE QUESTIONS IN PEN**
YOU MAY USE A CALCULATOR

Question 1

Answer the following True or False questions and give a short justification/counter-example:

a) If A is a square matrix with $nullity(A) = 0$, then A is the identity matrix. [2]

TRUE	
FALSE	

b) Every orthogonally diagonalisable matrix is invertible. [2]

TRUE	
FALSE	

c) If A has linearly independent eigenvectors $\mathbf{v}, \mathbf{w},$ then $\mathbf{v} \bullet \mathbf{w} = 0$. [2]

TRUE	
FALSE	

d) Let $\mathbf{v}, \mathbf{w}$ be vectors in an Inner Product Space V. If $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle$ for all vectors $\mathbf{u}$ in V, then $\mathbf{v} = \mathbf{w}$. [2]

TRUE	
FALSE	

e) The Gram-Schmidt process produces from a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ an orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ with the property that for each k, the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ span the same subspace as the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. [2]

TRUE	
FALSE	

f) Every linear system $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution. [2]

TRUE	
FALSE	

Question 2

Let $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ be an invertible matrix. Define an Inner Product on $\mathbb{R}^2$ as follows:

For all $\mathbf{x},\mathbf{y}$ in $\mathbb{R}^2$, let $\langle \mathbf{x},\mathbf{y} \rangle = \mathbf{x}^T A^T A \mathbf{y} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} A^T A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

a) Find $\langle (2,1),(-8,3) \rangle$. [3]

b) Find $\|(5,-2)\|$. [2]

Question 3

a) Express the Quadratic form $3x^2 + 4xy + 6y^2$ in matrix notation, i.e. find a symmetric matrix A that $\mathbf{x}^T A \mathbf{x}$ represents the quadratic form. [1]

b) Identify the curve represented by the conic $3x^2 + 4xy + 6y^2 - 1 = 0$ by determining whether the matrix, A, found in a) is positive definite, negative definite or neither. [2]

c) Use a rotation of axes to put the conic $3x^2 + 4xy + 6y^2 - 1 = 0$ in standard position. [5]

<u>Question 4</u>

Suppose $T:\mathcal{P}_2 \to \mathbb{R}^2$ the transformation defined by $T\left(p(x)\right)=\begin{bmatrix} p'(1) \\ \int_0^1 p(x)dx \end{bmatrix}$

a) Show that T is linear. [2]

b) Let $B=\left\{x^2+1, x+1, x-1\right\}$ be a basis for $\mathcal{P}_2$ and $C=\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\}$ be a basis for $\mathbb{R}^2$.

Find the matrix of T relative to the bases B and C. [4]

c) Using your answer in b) find $\left[T\left(x^2+2x+1\right)\right]_C$ and check your answer using T directly. [4]

Question 5

Let $T:\mathbb{R}^2 \to M_{22}$ be a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)=\begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$.

a) Determine a basis for the range of T. [2]

b) Determine the rank of T. [1]

c) Determine the nullity of T . [1]

d) Explain whether T is one-to-one or not. [1]

Question 6

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)=\begin{bmatrix} -1 \\ 2 \end{bmatrix}$,

$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)=\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

a) Find the standard matrix of the given transformation. [1]

b) Find $T(\mathbf{v})$ where $\mathbf{v}=\begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$. [1]

c) Is T onto? Explain. [2]

Question 7

Show that the linear transformation $T : M_{22} \to \mathcal{P}_3$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax^3 + bx^2 + cx + d$

is an isomorphism. [4]

Question 8

Consider the linear transformations $T : \mathcal{P}_3 \to \mathcal{P}_2$ and $S : \mathcal{P}_2 \to \mathcal{P}_3$ defined by the formulas

$T\left(p(x)\right) = p'(x)$, i.e. $T\left(ax^3 + bx^2 + cx + d\right) = 3ax^2 + 2bx + c$, and $S\left(p(x)\right) = xp(x)$, i.e. $S\left(ax^2 + bx + c\right) = ax^3 + bx^2 + cx$.

a) Find the inverse of S if it exists or explain why it does not exist. [2]

b) Show that $T \circ S \neq S \circ T$. [3]

Question 9

Let $T : V \rightarrow W$ be a linear transformation from the vector space V to the vector space W. Prove the following theorems.

a) The kernel T of is a subspace of V. [3]

b) T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$. [4]